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Separable systems in classical and wave mechanics.

Von

B. S. Madhava Rao in Bangalore (Indien).

1. Introduction.

The problem of integration of the classical Hamilton-Jacobi equation has been solved by the researches of Stäckel[1]), Levi-Civita[2]), Burgatti[3]) and Dall'Acqua[4]). Of the types where such separation is possible, the two most important are Stäckel's case and Levi-Civita's "essentially geodetic" case. Coming now to the question of solution of Schrödinger's wave equation which corresponds to the classical H-J[4a]) equation, it is well-known, as pointed out by Brillouin[5]), that separability of the latter does not necessarily imply that of the former. Some general separable types of the wave equation including a large number of cases met with in physical problems have also been given by Brillouin[6]). In the case where the H-J equation is separable under Stäckel's theorem, it has further been shown by Robertson[7]) that the separability of the corresponding wave equation involves an additional condition in the form of a functional identity.

I propose in this paper to give an alternative derivation of Robertson's condition by making use of the methods of Burgatti and Dall'Acqua and show that all the types given by Brillouin come under Stäckel's case and satisfy Robertson's condition. It is also shown that this condition is identically satisfied in the case of two degrees of freedom. An example is then constructed of a wave equation which is separable under Robertson's condition but does not come under Brillouin's types. I further prove that when the H-J equation is separable under Levi-Civita's geodetic case, the corresponding wave equation cannot be solved, except in trivial special cases, by separation of variables.

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1) P. Stäckel, Math. Annalen 42 (1893), S. 537.

2) T. Levi-Civita, Math. Annalen 59 (1904), S. 383.

3) P. Burgatti, Rom. Acc. Linc. Rend. (5) 20 (1911), p. 108.

4) F. A. Dall'Acqua, Math. Annalen 66 (1909), S. 398 and also Rend. Circ. Mat. Pal. 33 (1912), p. 341.

4a) H-J = Hamilton-Jacobi.

5) L. Brillouin, C. R. 183 (1926), p. 24.

6) Le Journ. d. Phys. et le Radium 7 (1926), p. 353.

7) H. P. Robertson, Math. Annalen 98 (1928), S. 749.

2. Robertson's condition for the Stäckel type.

Let us consider a mechanical system whose Hamiltonian is describable in terms of canonical co-ordinates and momenta and does not contain the time explicitly. Since there is no loss of generality, we might well consider the case of a single particle and put its mass equal to unity. Let the line element in the generalised q-co-ordinate space be given by

$$d s^2 = \sum_{r=1}^{n} \sum_{s=1}^{n} g_{rs} d q_r d q_s, \qquad (g_{rs} = g_{sr})$$

and let g denote the determinant formed by the g_{rs} and let $D = \sqrt{|g|}$. The elements of the determinant reciprocal to g are represented by the coefficients of inertia g^{rs} in the usual notation. The kinetic energy can be represented by

$$2T = \Sigma \Sigma g_{rs} \dot{q}_r \dot{q}_s$$

or, in the reciprocal form,

$$2K = \Sigma \Sigma g^{rs} p_r p_s$$

and the classical H-J equation can be written as

$$H = \tfrac{1}{2} \Sigma \Sigma g^{rs} p_r p_s + V = E \tag{1}$$

where V is a function of the co-ordinates only. In the case where (1) is separable under Stäckel's theorem we have $g^{rs} = 0\,(r \neq s)$ i. e. $g_{rs} = 0$ $(r \neq s)$ and it is possible to find a set of n^2 functions $\varphi_{ij}(q_i)$ and another set of n functions $\chi_i(q_i)$ such that

$$g^{rr} = \varphi^{r1}$$

and,

$$V = \sum_r \varphi^{r1} \chi_r(q_r)$$

where φ^{ij} is the general element of the determinant reciprocal to the determinant $\varphi = |\varphi_{ij}|$. The solution by separation of equation (1) then consists, according to Burgatti and Dall'Acqua [8]), of the system of equations

$$\begin{aligned} \tfrac{1}{2} p_r^2 &= \alpha_1 \varphi_{r1} + \alpha_2 \varphi_{r2} + \ldots + \alpha_{n-1} \varphi_{r,n-1} + E \varphi_{rn} - \chi_r \\ &= \sum_{j=1}^{n} (\alpha_j \varphi_{rj}) - \chi_r; \quad (r = 1, 2, \ldots, n;\ E = \alpha_n) \end{aligned} \tag{2}$$

such that the elimination of the arbitrary constants $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ between the equations (2) leads to a separable H-J equation.

Considered from the point of view of wave mechanics, equations (2) are to be treated as equations in operators operating on a wave function ψ and it is evident that when the Schrödinger equation corresponding to (1) is separable by writing $\psi = \psi_1 \psi_2 \ldots \psi_n$, the results of these operations will be ordinary differential equations for the different ψ_r's.

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[8]) F. A. Dall'Acqua, Reference [4]); see § 17 of the Pal. Rend. paper cited.

By finding the correct form for the operator p_r^2, I proceed to show that the separation of the wave equation involves the additional condition due to Robertson.

By writing Schrödinger's wave equation in the form

$$\text{div grad } \psi + 2k^2(E - V)\psi = 0 \qquad (k = 1/\hbar)$$

and working out the vectorial operators, divergence and gradient in the curved space, we obtain

$$\sum \sum \frac{1}{D} \frac{\partial}{\partial q_r} \left(D g^{rs} \frac{\partial \psi}{\partial q_s} \right) + 2k^2(E - V)\psi = 0 \tag{3}$$

and the density function is given by

$$P = D \psi \psi^*.$$

An operator F is Hermitian when

$$\int D \psi^* (F \psi) \, dq = \int D (F \psi)^* \psi \, dq$$

and the momentum-operator p_r is to be Hermitian in this sense and should satisfy the commutation condition

$$p_r q_r - q_r p_r = \frac{h}{i}.$$

We can therefore write [9])

$$p_r \psi = \frac{\hbar}{i} \frac{1}{\sqrt{D}} \frac{\partial}{\partial q_r} (\sqrt{D} \psi). \tag{4}$$

It is to be observed that equation (3) can also be immediately obtained from (1) if we write the latter in the correct form from the point of view of quantum mechanics. Such a form is well-known [10]) to be

$$H = \frac{1}{2} \sum \sum \frac{1}{\sqrt{D}} p_r D g^{rs} p_s \frac{1}{\sqrt{D}} + V = E. \tag{5}$$

If we now substitute in (5) the values for the operators p_r and p_s as given by (4) we get the wave equation in the form (3). We can now deduce from (5) the correct form for the operator p_r^2. Since

$$g^{rs} p_r p_s \text{ is to be replaced by } \frac{1}{\sqrt{D}} p_r D g^{rs} p_s \frac{1}{\sqrt{D}},$$

it follows that $p_r p_s = (g^{rs})^{-1} g^{rs} p_r p_s$ is to be replaced by

$$-\frac{1}{k^2} (g^{rs})^{-1} \frac{1}{\sqrt{D}} \frac{1}{\sqrt{D}} \frac{\partial}{\partial q_r} \sqrt{D} D g^{rs} \frac{1}{\sqrt{D}} \frac{\partial}{\partial q_s} \sqrt{D} \frac{1}{\sqrt{D}}$$

$$= -\frac{1}{k^2} (g^{rs})^{-1} \frac{1}{D} \frac{\partial}{\partial q_r} D g^{rs} \frac{\partial}{\partial q_s}.$$

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[9]) See W. Pauli, Handbuch der Phys. (Geiger-Scheel), 2te Aufl. (1933) **24** [1], Kap. 2, Ziff. 5, S. 120.

[10]) See B. Podolsky, Phys. Rev. **32** (1928), p. 812.

Hence,

$$p_r^2 = -\frac{1}{k^2}\frac{1}{g^{rr}}\frac{1}{\sqrt{g}}\frac{\partial}{\partial q_r}\sqrt{g}\,g^{rr}\frac{\partial}{\partial q_r}$$

and equations (2) can be written as

$$-\frac{1}{2k^2}\frac{1}{\sqrt{g}\,g^{rr}}\frac{\partial}{\partial q_r}\left(\sqrt{g}\,g^{rr}\frac{\partial}{\partial q_r}\right) = \sum_{j=1}^{n}(\alpha_j\,\varphi_{rj}) - \chi_r \quad (r = 1, 2, \ldots, n).$$

If equation (3) is to be separable by putting

$$\psi = \psi_1\,\psi_2 \cdots \psi_n$$

we should be able to operate, with both sides of the equation obtained above, on the wave function ψ_r. Hence

$$\frac{1}{\sqrt{g}\,g^{rr}}\frac{\partial}{\partial q_r}\left(\sqrt{g}\,g^{rr}\frac{\partial\,\psi_r}{\partial\,q_r}\right) + 2\,k^2\left[\sum_{j=1}^{n}(\alpha_j\,\varphi_{rj}) - \chi_r\right]\psi_r = 0. \tag{6}$$

Further in order that these be ordinary differential equations enabling the solution of the wave equation by separation, we should have

$$\frac{1}{\sqrt{g}\,g^{rr}}\frac{\partial}{\partial q_r}(\sqrt{g}\,g^{rr}) = \text{function of } q_r \text{ alone},$$

i. e.,

$$\frac{\partial}{\partial q_r}\{\log(\sqrt{g}\,g^{rr})\} = \text{function of } q_r \text{ alone}.$$

Assuming that the coefficients of inertia satisfy Stäckel's condition[11]), we can write $g^{rr} = \varphi^{r1}$ and the above condition reduces to

$$\varphi^{r1} = \frac{1}{\sqrt{g}}\,a_r F_r(q_r) \text{ where } a_r \text{ is independent of } q_r$$

i. e.,

$$\varphi\,\varphi^{r1} = \frac{1}{\sqrt{g}}\,\varphi\,a_r F_r.$$

But $\varphi\,\varphi^{r1}$ is the minor of φ_{r1} in the determinant φ and hence independent of q_r. Thus

$$\frac{1}{\sqrt{g}}\,\varphi = \frac{b_r}{a_r}\frac{1}{F_r}, \qquad b_r \text{ having the same significance as } a_r.$$

Since this condition must be true for all the co-ordinates $q_1 \ldots q_n$, we should have

$$\frac{1}{\sqrt{g}}\,\varphi = \prod_r\left(\frac{1}{F_r}\right) \tag{7}$$

which is exactly the condition obtained by Robertson.

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[11]) This is equivalent to assuming that when the wave equation is separable the same is true of the H-J equation, a statement which can easily be justified on general grounds.

Equations (6) now reduce to

$$F_r \frac{d\psi_r}{dq_r} + \frac{d^2\psi_r}{dq_r^2} + 2k^2\left[\sum_{j=1}^{n}(\alpha_j \varphi_{rj}) - \chi_r\right]\psi_r = 0$$

or

$$\frac{1}{f_r}\frac{d}{dq_r}\left(f_r \frac{d\psi_r}{dq_r}\right) + 2k^2\left[\sum_{j=1}^{n}(\alpha_j \varphi_{rj}) - \chi_r\right]\psi_r = 0 \tag{8}$$

and these are equations (9) of Robertson's article referred to under foot-note [7]).

3. Particular Cases.

Brillouin has considered three general types where the coefficients of inertia are given as the product of several functions, each function depending on only one co-ordinate.

Case 1:

$$g^{rr} = \mu_1(q_1)\,\mu_2(q_2)\ldots\mu_r(q_r); \qquad g^{rs} = 0 \quad (r \neq s)$$

and

$$V = \sum_r \mu_1(q_1)\,\mu_2(q_2)\ldots\mu_{r-1}(q_{r-1})\,\chi_r(q_r).$$

That this comes under Stäckel's type from the classical point of view is shown immediately by writing the determinant φ in the form

$$\varphi = \begin{vmatrix} \frac{1}{\mu_1}, & 0, & 0, & 0, & \ldots, & 0, & 0 \\ -1, & \frac{1}{\mu_2}, & 0, & 0, & \ldots, & 0, & 0 \\ 0, & -1, & \frac{1}{\mu_3}, & 0, & \ldots, & 0, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & 0, & 0, & \ldots, & -1, & \frac{1}{\mu_n} \end{vmatrix}.$$

The value of φ is $\mu_1\mu_2\ldots\mu_n$ and $g = \sqrt{g^{11}\ldots g^{nn}}$ and hence

$$\frac{1}{\sqrt{g}}\varphi = \prod_r (\mu_r/\sqrt{g^{rr}}) = \Pi\,(1/F_r)$$

and thus condition (7) is satisfied ensuring the separability of the wave equation.

Case 2:

$$g^{rr} = \mu_1(q_1)\,\mu_2(q_2)\ldots\mu_{r-1}(q_{r-1})\,\lambda_r(q_r),$$

$$V = \sum_r \mu_1(q_1)\,\mu_2(q_2)\ldots\mu_{r-1}(q_{r-1})\,\chi_r(q_r).$$

This can be at once reduced to the previous case by the change of variables

$$q_r' = \int \sqrt{(\lambda_r/\mu_r)}\,dq_r \qquad (r = 1, 2, \ldots, n).$$

Case 3: $$g^{rr} = \mu_1 \mu_2 \dots \mu_r \qquad (r \leqq l),$$

$$g_j^{l+\alpha,\, l+\alpha} = \mu_1 \mu_2 \dots \mu_l \mu_{j,\, l+1} \dots \mu_{j,\, l+\alpha},$$

$$V = \sum_{r=1}^{l} g^{r-1,\, r-1} \chi_r + \sum_{j=0}^{p} \sum_{\alpha=1}^{s_j} g_j^{l+\alpha-1,\, l+\alpha-1} \chi_{j,\, l+\alpha}.$$

This is the case of divergent branching and similar expressions can be written for the case of convergent branching. In the former case, the variables are denoted after the l^{th} by two indices and it comes under the generalisation of Stäckel's case given by Stäckel[12]) himself and obtained by replacing the system of variables

$$q_1, q_2, \dots, q_n \quad \text{by} \quad q_{k\varrho_k} \qquad (k = 1, 2, \dots, n;\ \varrho_k = 1, 2, \dots, h_k).$$

From the forms of g^{rr} $(r \leqq l)$ and $g_j^{l+\alpha,\, l+\alpha}$ as the product of functions, each of one variable, it is evident that condition (7) is satisfied.

Case 4: $n = 2$.

In the case of two degrees of freedom,

$$g^{11} = \varphi^{11}; \quad g^{22} = \varphi^{21}$$

and,

$$\frac{1}{\sqrt{g}}\, \varphi = \sqrt{g^{11} g^{22}}\, \varphi = \sqrt{\varphi\, \varphi^{11}} \cdot \sqrt{\varphi\, \varphi^{21}};$$

but $\varphi\, \varphi^{11}$ and $\varphi\, \varphi^{21}$ are the minors of φ_{11} and φ_{21} in the determinant φ and are therefore functions of q_2 and q_1 respectively. Hence Robertson's condition is identically satisfied and we can state the theorem that it is only the case of two degrees of freedom in which the separability of the H-J equation under Stäckel's condition implies the separability of the corresponding wave equation[13]).

Case 5: *Case* of *Stankewitch*.

In the first three cases considered above condition (7) is satisfied in virtue of the coefficients of inertia being products of functions each of one variable and the potential energy being a sum of such functions. I have been able to find an example which does not come under these three types but satisfies condition (7). Such an example is afforded by the mechanical system considered by Stankewitch[14]) where

$$T = \frac{1}{2} \sum_{r=1}^{n} \frac{K_r}{F_r(q_r)}\, \dot{q}_r^2,$$

$$V = \sum_{r=1}^{n} \chi_r(q_r)/K_r$$

[12]) P. Stäckel, C. R. **121** (1895), p. 489.

[13]) Cf. also, K. F. Herzfeld, Zeitschr. f. Phys. **68** (1931), S. 325, where this theorem is proved by direct verification.

[14]) J. Stankewitch, Mosk. Phys. Section **12** (1904), Lief. 1, p. 1—3. I have not been able to see the original article. For a summary, however, see F. d. M. **35** (1904), S. 719.

and further

$$K_r = (L_r - L_1)(L_r - L_2) \ldots (L_r - L_{r-1})(L_r - L_{r+1}) \ldots (L_r - L_n)$$

where

$$L_r = L_r(q_r).$$

To show that this comes under Stäckel's type consider the determinant φ given by

$$\varphi = \begin{vmatrix} L_1^{n-1}/F_1, & \ldots, & L_r^{n-1}/F_r, & \ldots, & L_n^{n-1}/F_n \\ 1/F_1, & \ldots, & 1/F_r, & \ldots, & 1/F_n \\ L_1/F_1, & \ldots, & L_r/F_r, & \ldots, & L_n/F_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ L_1^{n-2}/F_1, & \ldots, & L_r^{n-2}/F_r, & \ldots, & L_n^{n-2}/F_n \end{vmatrix}$$

which gives

$$\varphi = \frac{\pm 1}{F_1 F_2 \ldots F_n} \prod_{i,j=1}^{n} (L_i - L_j) \qquad (i < j).$$

Also $\varphi\, \varphi^{r1}$ = minor of φ_{r1} in φ

$$= \begin{vmatrix} 1/F_1, \ldots, 1/F_{r-1}, 1/F_{r+1}, \ldots, 1/F_n \\ \ldots \\ \ldots \end{vmatrix}$$

$$= \frac{\pm 1}{F_1 F_2 \ldots F_{r-1} F_{r+1} \ldots F_n} \prod_{h,k}^{n} (L_h - L_k) \qquad \begin{pmatrix} h, k \neq r \\ h < k \end{pmatrix}$$

and,

$$\varphi^{r1} = \frac{F_r \Pi (L_h - L_k)}{\Pi (L_i - L_j)} = \frac{F_r}{K_r} = \frac{1}{g^{rr}} = g^{rr}.$$

Thus

$$g^{rr} = \varphi^{r1}$$

and

$$V = \Sigma\, \varphi^{r1} \chi_r$$

which show that the present case belongs to Stäckel's type and from the form of K_r it is evident that it is not of Brillouin's type. Again the value of $1/\sqrt{g}$ is given by

$$\frac{1}{\sqrt{g}} = \frac{1}{\sqrt{g_{11} \ldots g_{nn}}} = \sqrt{\frac{F_1 F_2 \ldots F_n}{K_1 K_2 \ldots K_n}}.$$

Substituting the values for $K_1, \ldots, K_n$ in terms of L and multiplying, we easily find that

$$K_1 K_2 \ldots K_n = \Big(\prod_{i,j=1}^{n} (L_i - L_j) \Big)^2 \qquad (i < j)$$

and hence

$$\frac{1}{\sqrt{g}}\, \varphi = \frac{1}{\sqrt{F_1 F_2 \ldots F_n}},$$

proving that condition (7) is satisfied. We can thus state the theorem that the wave equation which corresponds to the system considered by Stankewitch can be solved by separation. I have not so far been able to find in current physical literature a wave equation of this type.

4. Levi-Civita's geodetic case.

In this case the classical H-J equation is given by

$$H = \tfrac{1}{2} \sum \sum g^{rs} p_r p_s = E \tag{9}$$

since it is not possible to associate a potential function with the equation, $H = E$. The solution by separation of equation (9) is given, according to Burgatti and Dall' Acqua, by

$$p_r = \alpha_1 \varphi_{r1} + \alpha_2 \varphi_{r2} + \ldots + \alpha_{n-1} \varphi_{r,n-1} + \varphi_{rn} \sqrt{2E - \tau} \tag{10}$$

where τ is a quadratic in $\alpha_1, \ldots, \alpha_{n-1}$ and φ_{ij} is a function of q_i alone. If the wave equation which can be set up from (9) can be solved by separation of variables, the solution must consist of the differential equations obtained from (10) by operating on the separate wave functions $\psi_1, \ldots, \psi_n$. Using for the momentum-operator p_r the expression (4) given by Pauli and operating on ψ_r we have

$$\frac{h}{i} \frac{1}{\sqrt{D}} \frac{\partial}{\partial q_r} (\sqrt{D} \psi_r) = \sum_{j=1}^{n-1} (\alpha_j \varphi_{rj}) + \varphi_{rn} \sqrt{2E - \tau}.$$

In order that this be an ordinary differential equation enabling the solution by separation, we should have

$$\frac{1}{\sqrt{D}} \frac{\partial}{\partial q_r} (\sqrt{D}) = \text{function of } q_r \text{ alone},$$

and this, being true for all the co-ordinates, is satisfied only if

$$g = \prod_{r=1}^{n} f_r(q_r). \tag{11}$$

We can consider the determinant g as obtained by bordering the minor of g_{11} by the elements of the first row and first column and expand the determinant in terms of products of these elements. Such an expansion would give

$$g = g_{11} \frac{\partial g}{\partial g_{11}} - \sum \sum g_{1r} g_{s1} \frac{\partial^2 g}{\partial g_{11} \partial g_{sr}} \qquad (r, s \neq i).$$

Since by an interchange of rows and columns any g_{ii} can be made the leading diagonal element we should similarly have

$$g = g_{ii} \frac{\partial g}{\partial g_{ii}} - \sum \sum g_{ir} g_{si} \frac{\partial^2 g}{\partial g_{ii} \partial g_{sr}} \qquad (r, s \neq i). \tag{12}$$

In the essentially geodetic case it is well-known[15]) that any element g_{rs} is a function only of q_r and q_s and of the form $g_{rs} = \sum_{j=1}^{n} a_r^{(j)} a_s^{(j)}$ where a_r and a_s are functions respectively of q_r and q_s such that

$$\sum_j a_r^{(j)2} = \sum_j a_s^{(j)2} = 1.$$

It follows therefore that the partial derivatives appearing in (12) are independent of q_i and equation (11) can be satisfied only if either (a) $g_{ir} = 0$ $(i \neq r)$ or (b) $g_{ir} g_{si}$ contains g_{ii} as a factor. In the latter case it follows from the form of g_{ir} that all the functions $a_i^{(j)}$ must be of the form $A_j \sqrt{g_{ii}}$ where A_j is a constant, and similarly g_{ir} must also contain $\sqrt{g_{rr}}$ as a factor. From the equation $\Sigma a_i^{(j)2} = 1$ we can therefore conclude that g_{ii} must be a constant. The same is true for g_{rr} and g_{ss} which means that g_{ir} and g_{si} are constants. Thus all the coefficients of inertia are constants and the Hamiltonian is such that all the q's are cyclic co-ordinates and we no longer have the geodetic case. The other alternative (a) that $g_{ir} = 0$ $(i \neq r)$ reduces g to $g_{11} \ldots g_{nn}$ and condition (11) is no doubt satisfied, but the Hamiltonian comes under the completely separable case. Hence, neglecting the trivial cases of all co-ordinates being cyclic and of complete separability, we can state the theorem that the wave equation cannot be solved by separation in the essentially geodetic case.

The non-separability of the wave equation in this case can also be understood by means of the geometrical significance of separation variables[16]). The essence of the possibility of separation consists in this that, when one considers all eigenvibrations, the nodal surfaces split up into n groups such that the totality of surfaces of a group do not cut each other and lie everywhere dense. These groups then define the separation variables of Schrödinger's wave equation. In the present geodetic case the co-ordinate n-surfaces are *surfaces of translation* and the same is true of the groups of nodal surfaces. From the method of generation of an n-surface of translation by imposing on a $(n-1)$-surface translations congruent with a set of other similar surfaces, it can be seen that, in general, surfaces of the same group intersect one another. Thus nodal surfaces of different solutions intersect in an irregular way so that the construction of the set of separation variables is not possible in this case.

This can be illustrated clearly in the case of two degrees of freedom by writing the linear element as

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2.$$

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15) See F. A. Dall' Acqua, Math. Annalen 66 (1909), S. 404—405, § 5.

16) See K. F. Herzfeld, Reference 13) § II.

In the geodetic case E is a function of u only and G a function of v only, and the line element can be reduced to

$$ds^2 = du^2 + 2\cos(U + V)\, du\, dv + dv^2,$$

U and V being functions respectively of u and v. But it is well-known [17]) that these are exactly the conditions for the parametric curves to form a *Tchebychef system* on a surface. From the well-known property that the tangent to either family of a Tchebychef system undergoes a parallel displacement along each curve of the other family, it can be seen that members of the same family intersect one another in general.

[Added in the proofs February 1935.] After this paper had been sent for publication, my attention was kindly drawn by Professor O. Blumenthal to a paper by L. P. Eisenhart on "*Separable systems of Stäckel*" in Annals of Math. **35**, 2 (1934), p. 284. In this paper Eisenhart derives the very interesting result that Robertson's condition is equivalent to the vanishing of the Ricci tensor for the space considered and hence satisfied for an Euclidean space or for a space of constant Riemannian curvature. He also derives the canonical forms, for the equations which are reducible in the case $n = 3$. It will be readily seen that Brillouin's types are included in the forms given by Eisenhart. Again, his general canonical form leading to a system of confocal quadrics affording separation of variables is identical with the case of Stankewitch (see § 3 above) which I have constructed as an example satisfying Robertson's condition.

In view of what I have proved in § 4 of my paper, Eisenhart's geometrical interpretation of Robertson's condition leads to an interesting observation. It is well-known that for separability of the H-J equation under Levi-Civita's type it is a *necessary* condition that the space be of the Euclidean variety (Cf. Levi-Civita, Math. Annalen **59**, S. 383). This is however not necessary for the Stäckel type and it now appears according to Eisenhart, that imposing this as an additional condition makes the wave equation also separable. This affords a sort of justification for my result that the wave equation corresponding to Levi-Civita's case cannot be solved by separation, for otherwise, it would mean a still further specialisation of a space already Euclidean.

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[17]) L. Bianchi, Lezioni di geometria differenziale. 3. Aufl. (1920), I, S. 153—162.

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In diesem Buche bin ich bestrebt, meine Überlegungen in demjenigen philosophischen Geiste zu halten, für den ich in den Untersuchungen Ludwig Wittgensteins das Vorbild zu finden glaube.

Viele Absätze dieses Buches gleichen Warnungstafeln und Wegweisern, die man leicht übersehen kann. Der Leser möge daher stets darauf achten, welchem Vorurteile in einem Absatz vorgebeugt wird oder zu welcher logischen Überlegung man hierdurch angeregt wird.

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